Asymptotic behavior of the solution of the two-dimensional stochastic vorticity equation

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Asymptotic properties of the solution of the vorticity equation for two-dimensional randomly stirred fluid with long-range correlations of the driving force are analyzed with the aid of field-theoretic renormalization group methods. Renormalization due to the force fluctuations is shown to lead to drastic changes in the relative contribution of microscale viscosity and macroscale friction to the energy and enstrophy dissipation. [S1063-651X(98)03610-1]

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I. INTRODUCTION

Since the late 1970s renormalization-group methods have been widely applied to the investigation of fully developed three-dimensional turbulence [1–3]. In this approach the stochastically forced Navier-Stokes equation has been used with powerlike falloff in the space of the correlations of the random forcing. Scaling exponents and amplitude coefficients describing the asymptotic behavior of various physical quantities have been calculated in an ε expansion (see, e.g., the recent review [4]). The expansion parameter $\varepsilon=2-\lambda$ is the deviation of the power of wave number λ in the correlation function of the random force from the critical value $\lambda_c=2$, at which the coupling constant of the corresponding field theory is dimensionless.

In these calculations the value of the expansion parameter ε has been chosen such that the only dimensional parameters of the model are the viscosity and the energy pumping rate.

From the renormalization-group equations it then follows that the asymptotic behavior of the model at large spatial scales is independent of the viscosity, and the powerlike wave-number dependence of the equal-time velocity-velocity correlation function is exactly that predicted by the Kolmogorov scaling law [5]. Therefore, the renormalization-group approach yields a mean-field description of turbulence with built-in Kolmogorov scaling, in which the ε expansion may be constructed, e.g., for the structure functions.

In the renormalization-group approach most work has been carried out in three dimensions. Only recently this approach has been applied to the analysis of two-dimensional turbulence [6,7]. There are both physical and technical reasons for this. In two-dimensional turbulence at the scales of the stirring length coherent vortex structures are created, which inhibit formation of self-similar structures. However, experimental data on atmospheric turbulence [8], and recent numerical simulations [9] indicate that in two-dimensional turbulence there may occur two scaling regimes corresponding to the inverse energy cascade towards small wave numbers and the enstrophy cascade towards large wave numbers.

The existence of these two scaling regimes is in accord with the prediction of Kraichnan [10]. It should be noted, however, that the energy (enstrophy) pumping leading to a steady state with the two scaling regimes may be realized in two different ways. In numerical simulations [9] and some experiments [11] the energy (enstrophy) pumping takes place on scales in between the inverse energy cascade and the enstrophy cascade. In the atmospheric turbulence [8] the energy and enstrophy sources are at the outer edges of the scaling intervals, and it is not clear whether there is an energy and enstrophy sink between them [8] or they coexist [9]. In both cases the Kolmogorov spectrum of the inverse energy cascade $E(k)\propto k^{-5/3}$ for $k\ll k_I$ is observed experimentally and in the simulations (with the exception [12]) for wave numbers smaller than the inverse length scale of the energy pumping $k_I\propto 1/l_I$. However, in the numerical simulations [9] in the enstrophy inertial range $k\gg k_I$ the falloff of the energy spectrum seems to be steeper than $E(k)\propto k^{-3}$, predicted by dimensional arguments [10].

From the technical point of view it is not possible to use the renormalized d-dimensional model at two dimensions by simply putting $d=2$ in the results [2]. The reason is that in the two-dimensional case there is an additional class of divergent graphs, which have to be included in the renormalization procedure. The account of the contribution of these divergent graphs has led to significant confusion [6,13,14]. In particular, an incorrect renormalization of the two-dimensional stochastic vorticity equation has led to false conclusions about the asymptotic behavior of the solution of this equation [6].

The source of this confusion can be explained as follows. The correlation function of the random force, which is usually used in the description of turbulence, in the wave-number space is $\propto k^{4-d}(k^2+m^2)^{-\varepsilon}$ [1,15], where m is the small wave-number cutoff. For the ε expansion the cutoff parameter may be, and often has been, chosen $m=0$. However, for finite ε a careful analysis of the limit $m\to 0$ in this expansion is required. For arbitrary real d and ε, $k^{4-d}(k^2+m^2)^{-\varepsilon}$ is a singular function of k^2 at the origin in the limit $m\to 0$, which corresponds to long-range correlated random force. At two dimensions the correlation function is renormalized by counterterms $\propto k^2$, which correspond to local correlations in the coordinate space. The renormalization is carried out in the logarithmic model, in which $\varepsilon=0$. In two dimensions the original correlation function cannot be distinguished from the local counterterms $\propto k^2$, and it is not obvious how the model should be renormalized. In particular, to prescribe the local in space counterterms to renormalization of the nonlocal (in the limit $m\to 0$) correlation function [6,13] is not a consistent way to renormalize the model.

Recently, a renormalization procedure has been put forward [16], in which the renormalization of the correlation function of the random force has been carried out in a consistent manner for d-dimensional turbulence near two dimensions. In the present two-dimensional case the argument is the following. In order to deal with finite quantities, the model must be regularized. To this end the analytic regularization with the parameter ε must be used, otherwise there is no way to distinguish between the original correlation function $\propto k^{2-2\varepsilon}(1+m^2/k^2)^{-\varepsilon}$ and the local counterterms $\propto k^2$. The renormalization is most conveniently carried out by multiplicative renormalization. In order to make the model multiplicatively renormalizable the local term $\propto k^2$ is added to the force correlation function at the outset. Only this term is then renormalized, whereas the nonlocal term is left intact, contrary to the earlier treatment of the d-dimensional model near two dimensions [6,13,14].

In the present paper this renormalization procedure is applied to the solution of the two-dimensional stochastic vorticity equation. The stochastic problem and the subsequent field theory are used in the form specific of two-dimensional turbulence, i.e., the starting point is the stochastic vorticity equation for the stream function ψ instead of the Navier-Stokes equation for the two-dimensional solenoidal velocity field $\mathbf{v}=(\mathbf{e}_z\times\nabla)\psi$.

The paper is organized as follows. In Sec. II the correct renormalization of the field theory corresponding to the stochastic vorticity equation is carried out and the renormalization constants calculated at one-loop order. In Sec. III the renormalization-group equations are set up and fixed points found. In Sec. IV the asymptotic energy spectrum is calculated for the energy and enstrophy inertial ranges. Section V is devoted to a discussion and concluding remarks. Details of the calculation of the renormalization constants are presented in the Appendix.

II. RENORMALIZATION OF THE SOLUTION OF THE STOCHASTIC VORTICITY EQUATION

Consider the stochastic Navier-Stokes equation for the flow of homogeneous incompressible fluid, which for the transverse components of the velocity field assumes the form

$$\partial_t v_i + P_{ij} v_l \partial_l v_j = \nu_0 \nabla^2 v_i - \xi_0 v_i + f_i, \quad \partial_i v_i = 0. \tag{1}$$

Here $v_i(t,\mathbf{x})$ are the coordinates of the divergenceless velocity field, ν_0 is the kinematic viscosity, ξ_0 is the coefficient of friction, and P_{ij} is the transverse projection operator ($P_{ij} = \delta_{ij} - k_i k_j/k^2$ in the wave-number space), and f_i are the coordinates of the random force. Here and henceforth, summation over repeated indices is implied. The drag term is added to the Navier-Stokes equation because in experimental realizations and simulations of a two-dimensional turbulent flow energy is consumed not only by microscale dissipation, but also by the friction at the boundaries of the fluid layer. The drag force makes it possible to maintain a stationary state with the anticipated inverse energy cascade towards small wave numbers and the enstrophy (squared vorticity) cascade towards large wave numbers.

In the applications of the stochastic Navier-Stokes equation (1) to turbulence the random force is assumed to have a Gaussian distribution with zero mean and the correlation function in the wave-vector space [1–3] of the form

$$\langle f_i(t,\mathbf{k}) f_j(t',\mathbf{k}')\rangle = P_{ij}\delta(\mathbf{k}+\mathbf{k}')\,\delta(t-t')\bar{D}(k). \tag{2}$$

Here,

$$\bar{D}(k) = g_0 \nu_0^3 \frac{k^{4-d}}{(k^2+m^2)^{2-\lambda}}, \tag{3}$$

and λ is an arbitrary parameter.

The force correlation function is related to two basic physical quantities, the energy pumping rate $\mathcal{E}$ and the enstrophy pumping rate $\mathcal{B}$ as

$$\mathcal{E} = \frac{d-1}{2}\int \frac{d\mathbf{k}}{(2\pi)^d}\bar{D}(k), \quad \mathcal{B} = \frac{d-1}{2}\int \frac{d\mathbf{k}}{(2\pi)^d} k^2 \bar{D}(k) \tag{4}$$

in d dimensions, which allows one to connect the ‘‘coupling constant’’ g_0 with the pumping rate in the corresponding asymptotic region.

The correlation function (3) is chosen to be locally integrable (due to the IR cutoff m^2) and to have a powerlike falloff characterized by the exponent λ in the wave-vector space. The large wave-number behavior is essential for consistent renormalization, whereas the small wave-number behavior follows the three-dimensional tradition [1]. When the particular function (3) is used, an additional UV cutoff Λ is required for large enough values of the exponent λ. The value of the falloff exponent determines whether the energy (enstrophy) pumping is concentrated at large or small wave numbers. The ‘‘physical’’ value of the falloff exponent is determined by the condition that the energy (enstrophy) spectrum in the inertial range is independent of the cutoff parameters in the wave-vector space.

In the two-dimensional case it is customary to express the velocity field using the stream function ψ defined by the relation $v_i = \epsilon_{ij}\partial_j\psi$, where ϵ_{ij} is the second-rank antisymmetric tensor with the usual normalization $\epsilon_{12}=1$. In two dimensions the vorticity $\omega = \nabla\times\mathbf{v}$ is related to the stream function as $\omega = \epsilon_{ij}\partial_i v_j = -\nabla^2\psi$.

Two-dimensional curl of the Navier-Stokes equation (1) yields the vorticity equation. When the velocity is expressed in terms of the stream function the vorticity equation assumes the form

$$\partial_t \nabla^2\psi + \partial_i\partial_m(\epsilon_{mn}\partial_n\psi\partial_i\psi) = \nu_0\nabla^4\psi - \xi_0\nabla^2\psi + f, \tag{5}$$

where $f = -\epsilon_{ij}\partial_i f_j$. The correlation function of the random force here is

$$\langle f(t,\mathbf{k}) f(t',\mathbf{k}')\rangle = \delta(\mathbf{k}+\mathbf{k}')\,\delta(t-t')D(k), \tag{6}$$

where

$$D(k) = g_0\nu_0^3 \frac{k^4}{(k^2+m^2)^{2-\lambda}}, \tag{7}$$

It should be noted that the (nonstochastic) vorticity equation (5) is Galilei invariant, whereas the two-dimensional Navier-Stokes equation with the drag term (1) is not.

The stochastic problem (5), (6) may be cast [1,2] in a field theory with the ‘‘action’’

$$S=\frac{1}{2}\psi' D\psi' + \psi' \times[-\partial_t\nabla^2\psi+\nu_0\nabla^4\psi-\partial_i\partial_m(\epsilon_{mn}\partial_n\psi\partial_i\psi)-\xi_0\nabla^2\psi]. \tag{8}$$

Here, all the necessary integrals and sums are implied. It is convenient to assign canonical scaling dimensions to the parameters of the action (8) separately with respect to wave number (d^k) and frequency (d^ω) variables with the convention $d_k^k=-d_x^k=1$, $d_\omega^\omega=-d_t^\omega=1$. The total dimension of a parameter P is defined as $d_P=d_P^k+2d_P^\omega$. The canonical dimensions are determined from the condition that the action (8) is scale invariant with respect to spatial coordinates and time separately. Thus,

$$\begin{aligned}
&d_\psi^k=-2,\; d_\psi^\omega=1,\quad d_\psi=0;\\
&d_{\psi'}^k=2,\; d_{\psi'}^\omega=-1,\quad d_{\psi'}=0;\\
&d_{\nu_0}^k=-2,\; d_{\nu_0}^\omega=1,\quad d_{\nu_0}=0;\\
&d_{\xi_0}^k=0,\; d_{\xi_0}^\omega=1,\quad d_{\xi_0}=2;\\
&d_{g_0}^k=4-2\lambda,\; d_{g_0}^\omega=0,\quad d_{g_0}=4-2\lambda.
\end{aligned} \tag{9}$$

The theory is logarithmic, i.e., $d_{g_0}=0$, when $\lambda=2$.

Power counting in the graphs shows that the logarithmic model is renormalizable in spite of the vanishing scaling dimensions of the fields. Due to the definition of the stream function, there are enough factorizing external wave vectors at the interaction vertex to keep the model renormalizable: a linear wave vector for each ψ argument and a quadratic in the wave-vector coordinates term for each ψ' argument of one-particle-irreducible (1PI) Green functions. Due to this, when the model is logarithmic ($\lambda=2$), the real degree of divergence of a 1PI Green function is $\delta'=4-n-2n'$, where n and n' are the numbers of the ψ and ψ' arguments, respectively.

As a consequence of the Galilei invariance of the action (8), the 1PI Green function $\Gamma_{\psi\psi\psi'}$, which is superficially divergent by power counting, is actually convergent, as in the Navier-Stokes problem [1,17]. Therefore, only the graphs of the 1PI Green functions $\Gamma_{\psi\psi'}$ and $\Gamma_{\psi'\psi'}$ yield divergent contributions to the renormalization of the model.

It should be borne in mind that the renormalization is dealing with the UV divergences of the model. The renormalized model exhibits scale-invariant behavior in the limit governed by a stable fixed point of the renormalization group. In the present case there is an IR stable fixed point, which yields the self-similar behavior of the model in the limit of small wave numbers.

If the resulting renormalized model is finite in the limit $m\to 0$, then the self-similar behavior of the model is given by the solution of the renormalization-group equations. This is always the case in ε expansion. For finite ε additional analysis of the limit $m\to 0$ is required, which is not simple. Detailed discussion of these problems is deferred to Sec. IV.

The divergences brought about by $\Gamma_{\psi\psi'}$ may be absorbed in the renormalization of the parameter ν_0. The coefficient of friction ξ_0 is not renormalized at all due to the factorizing derivatives at the interaction vertex: in the graphs of $\Gamma_{\psi\psi'}$ cubic polynomials in the coordinates of the wave-vector $\mathbf{k}$ factorize, therefore the counterterms are $\propto k^4$.

Divergent terms are polynomial functions of the wave numbers in the regularized model [18]. However, in the bare Green function $\Gamma_{\psi'\psi'0}(\omega,\mathbf{k})=g_0\nu_0^3k^4(k^2+m^2)^{\lambda-2}$ there is no such term to be renormalized by the divergent contributions of $\Gamma_{\psi'\psi'}$. The reason is that in order to keep track of the long-range correlation function an analytic regularization must be used, e.g., with the parameter $\varepsilon=2-\lambda$, which is used here. Note that the difference between long-range and short-range correlations of the random field is meaningful only in the limit of vanishing inverse correlation length $m\to 0$. To keep the model multiplicatively renormalizable, a regular $\propto k^4$ term must be added to the correlation function at the outset. Hence, in the correlation function (6) the term $D(k)=g_0\nu_0^3k^4(k^2+m^2)^{\lambda-2}$ is replaced by the sum

$$D(k)=g_0\nu_0^3\frac{k^4}{(k^2+m^2)^{2-\lambda}}+a_0\nu_0^3k^4 \tag{10}$$

with a new parameter a_0.

The divergences of $\Gamma_{\psi'\psi'}$ give rise to the renormalization of the parameter $a_0\nu_0^3$, whereas the parameter $g_0\nu_0^3$ remains unchanged. The canonical scaling dimensions of the parameter a_0 are $d_{a_0}^k=d_{a_0}^\omega=d_{a_0}=0$. As a result, the renormalized action may be written as

$$\begin{aligned}
S_R=&\frac{1}{2}g\nu^3M^{2\varepsilon}\nabla^2\psi'(-\nabla^2+m^2)^{\lambda-2}\nabla^2\psi'\\
&+\frac{1}{2}Z_1a\nu^3\nabla^2\psi'\nabla^2\psi'+\psi'[-\partial_t\nabla^2\psi+Z_\nu\nu\nabla^4\psi\\
&-\partial_i\partial_m(\epsilon_{mn}\partial_n\psi\partial_i\psi)-\xi_0\nabla^2\psi],
\end{aligned} \tag{11}$$

where M is the renormalization mass, and the renormalized parameters are defined by

$$\nu_0=Z_\nu\nu,$$

$$a_0=aZ_a=aZ_1Z_\nu^{-3},$$

$$g_0=M^{2\varepsilon}gZ_g.$$

As usual, the renormalized coupling constants g and a are chosen to be both spatially and temporally dimensionless. The nonlocal term of the action (11) is not renormalized, therefore the renormalization constants Z_g and Z_ν are related as

$$Z_g=Z_\nu^{-3}, \tag{12}$$

up to a finite renormalization. In the minimal subtraction scheme [18] used here, the relation (12) holds as it stands.

In the minimal subtraction scheme only the singular contributions of the graphs to the renormalization constants are retained. In general, the renormalization constants are determined up to a finite renormalization, which may be used to relate the parameters of the model to observables at some reference scale. Here, a natural choice would be

$$\frac{1}{24}\frac{\partial^4}{\partial k^4}W^{-1}_{\psi\psi' R}(\omega,k)\big|_{\substack{\omega=M^2\nu\\ k=M}}=-\nu_0, \tag{13}$$

where $W_{\psi\psi' R}$ is the renormalized complete (dressed) response function of the stream function, and ν_0 is the viscosity at the reference wave number M. The normalization (13) implies that at wave numbers of the order of M the nonlinear terms are negligible.

The choice of renormalization scheme does not affect the scaling exponents, but it may change the scaling functions. However, in the ε expansion any renormalization prescription different from the minimal subtraction scheme changes the coefficient and scaling functions by terms that are of higher than the leading order in the ε expansion. In the present work the correlation functions and spectra are calculated at the leading order of the ε expansion, which is uniquely given by the minimal subtraction procedure.

Due to the addition of the parameter a the regularization prescription had to be changed. Practically the most convenient way to introduce an ultraviolet cutoff turned out to be a kind of Pauli-Villars regularization by the substitution

$$a\nu^3k^4 \to a\nu^3k^4\frac{\Lambda^2}{\Lambda^2+k^2}.$$

This implies that the renormalized model is obtained as the double limit of the regularized model, when $\varepsilon\to 0$ and $\Lambda\to\infty$. The result depends on the order of passing to the limit, but this ambiguity is no more dangerous than that related to finite renormalization, and thus does not affect the asymptotic behavior of the model.

For the divergent parts of the renormalization constants at the one-loop level I obtain

$$Z_1 = 1-\frac{1}{64\pi}\left\{\frac{g^2}{2\varepsilon a}+g\begin{bmatrix}\dfrac{2}{\varepsilon}, \\ 2\ln\dfrac{\Lambda^2}{M^2}\end{bmatrix}+a\ln\frac{\Lambda^2}{M^2}\right\}, \tag{14}$$

$$Z_\nu = 1-\frac{1}{64\pi}\left[\frac{g}{\varepsilon}+a\ln\frac{\Lambda^2}{M^2}\right].$$

Here, the upper expression for Z_1 corresponds to the limit in which first $\Lambda\to\infty$ and then $\varepsilon\to 0$, and the lower expression to the reversed order of passing to the limits.

III. RENORMALIZATION-GROUP EQUATIONS AND FIXED POINTS

From the connection between the renormalized and unrenormalized equal-time autocorrelation functions of the stream function ψ

$$W^{\mathrm{st}}_{\psi\psi R}(g,a,\nu,M) = W^{\mathrm{st}}_{\psi\psi}(g_0,a_0,\nu_0)$$

the usual basic renormalization-group equation follows:

$$[\mathcal{D}_M+\beta_g\partial_g+\beta_a\partial_a-\gamma_\nu\mathcal{D}_\nu]W^{\mathrm{st}}_{\psi\psi R}=0, \tag{15}$$

where

$$\gamma_i = \tilde{\mathcal{D}}_M \ln Z_i, \quad i=a,g,\nu,$$

$$\beta_g = \tilde{\mathcal{D}}_M g = g(-2\varepsilon-\gamma_g), \tag{16}$$

$$\beta_a = \tilde{\mathcal{D}}_M a = -a\gamma_a,$$

and

$$\tilde{\mathcal{D}}_M = M\partial_M|_0,$$

$$\mathcal{D}_M = M\partial_M, \tag{17}$$

$$\mathcal{D}_\nu = \nu\partial_\nu.$$

In $\tilde{\mathcal{D}}_M$ the partial derivative with respect to M is calculated with fixed bare parameters (subscript 0), whereas in $\mathcal{D}_M$ the renormalized parameters (without subscript) are kept fixed.

The relation (12) together with the definitions (17) leads to the connection [1]

$$\gamma_g = -3\gamma_\nu. \tag{18}$$

It is convenient to extract the function R of dimensionless arguments a, g, and

$$s=-\ln\frac{k}{M},\quad z=\frac{\xi_0}{M^2\nu},\quad u=\frac{m}{k}$$

from the correlation function $W^{\mathrm{st}}_{\psi\psi R}$:

$$W^{\mathrm{st}}_{\psi\psi R}(k,g,a,\nu,\xi_0,m,M) = \tfrac{1}{2}g\nu^2M^{2\varepsilon}k^{-2-2\varepsilon}R(e^{-s},g,a,z,u). \tag{19}$$

Substitution of this representation in the basic equation (15) leads to the following Callan-Symanzik equation for the function R:

$$[\partial_s-(2-\gamma_\nu)\mathcal{D}_z+\beta_g\partial_g+\beta_a\partial_a+\gamma_\nu]R(e^{-s},g,a,z,u)=0.$$

The solution of this first-order partial differential equation may be written in the form

$$R(e^{-s},g,a,z,u)=e^{-2\int_0^s\gamma_\nu d\sigma}R(1,\bar{g},\bar{a},\bar{z},u), \tag{20}$$

where $\bar{g}$, $\bar{a}$ are the solutions of the equations

$$\int_g^{\bar{g}}\frac{dg}{\beta_g}=-s,\quad \int_a^{\bar{a}}\frac{da}{\beta_a}=-s, \tag{21}$$

and

$$\bar{z}=ze^{\int_0^s(2-\gamma_\nu)d\sigma}. \tag{22}$$

Due to the connection (18) the exponential scaling factors in Eqs. (20) and (22) may be expressed in terms of the running coupling constants $\bar{a}$, $\bar{g}$ [15], and

$$W^{\mathrm{st}}_{\psi\psi R}(k,g,a,\nu,\xi_0,m,M) = \tfrac{1}{2}\bar{g}^{1/3}g^{2/3}\nu^2k^{-(2+4\varepsilon/3)}M^{4\varepsilon/3}R\left[1,\bar{g},\bar{a},z\left(\frac{k}{M}\right)^{-(2-2\varepsilon/3)}\left(\frac{\bar{g}}{g}\right)^{1/3},u\right]. \tag{23}$$

It is convenient to rewrite this relation in terms of the unrenormalized (physical) parameters:

$$W^{\rm st}_{\psi\psi R}(k,g,a,\nu,\xi_0,m,M)$$
$$=\tfrac{1}{2}\overline{g}^{1/3}g_0^{2/3}\nu_0^2k^{-(2+4\varepsilon/3)}R\left[1,\overline{g},\overline{a},\frac{\overline{g}^{1/3}\xi_0}{g_0^{1/3}\nu_0}k^{-(2-2\varepsilon/3)},u\right]. \tag{24}$$

One-loop calculation yields the following expressions for the β functions (16):

$$\beta_g=g'(-2\varepsilon+3a'+3g'),$$
$$\beta_a=-g'^2+g'a'+2a'^2, \tag{25}$$

where

$$a'=\frac{a}{32\pi},\quad g'=\frac{g}{32\pi}.$$

These β functions are exactly the same as those of the d-dimensional Navier-Stokes equation at two dimensions [16]. Thus, it seems that from the point of view of the renormalization group, the results of the d-dimensional model in the two-parameter expansion [16] may be applied directly to the two-dimensional case.

The fixed points are determined by the system of equations $\beta_g=\beta_a=0$. From the solution of the equations (21) near a fixed point it follows that the fixed point is infrared stable, when the matrix $\omega_{nm}=\partial_n\beta_m$ is positively definite at the fixed point. The trivial fixed point $g'_*=a'_*=0$ is infrared stable only if $\varepsilon<0$. For $\varepsilon>0$, the trivial fixed point is a saddle point of the solution of Eq. (21).

The anomalous asymptotic behavior of the model at small wave numbers is governed by the nontrivial fixed point

$$g'_*=\tfrac{4}{9}\varepsilon,\quad a'_*=\tfrac{2}{9}\varepsilon \tag{26}$$

at which the eigenvalues of the stability matrix are

$$\omega_{1,2}=\tfrac{2}{3}(2\pm i\sqrt{2})\varepsilon. \tag{27}$$

The real parts of both eigenvalues are positive, when $\varepsilon>0$ and this inequality determines the region of stability of this fixed point. Since the eigenvalues of the ω matrix are complex conjugate, the fixed point is an infrared-stable focus. The anomalous dimension γ_ν^* is related to the parameter ε in the usual manner [1,2] $\gamma_\nu^*=2\varepsilon/3$.

In the basin of attraction of an infrared-stable fixed point the running coupling constants approach the fixed-point values $\overline{g}\to g_*$, $\overline{a}\to a_*$ in the large-scale limit, when $s\to\infty$. However, the parameter z grows in this limit, as seen from Eq. (22), and there are two separate asymptotic limits corresponding to wave-number scales much greater or much smaller than the borderline wave number

$$k_b=\left[\frac{\xi_0^3}{\nu_0^3}\frac{g_*}{g_0}\right]^{1/(6-2\varepsilon)},\quad \varepsilon\neq 3, \tag{28}$$

at which both dissipative terms in the renormalized solution of the vorticity equation are of the same order of magnitude. In the former limit dissipation due to drag is small, in the latter dissipation due to microscale viscosity is small.

IV. ASYMPTOTIC ENERGY SPECTRUM

From the solution (24) the asymptotic expression

$$W^{\rm st}_{\psi\psi R}(k,g,a,\nu,\xi_0,m,M)$$
$$=\tfrac{1}{2}g_*^{1/3}g_0^{2/3}\nu_0^2k^{-(2+4\varepsilon/3)}R\left[1,g_*,a_*,\left(\frac{k_b}{k}\right)^{2-2\varepsilon/3},\frac{m}{k}\right] \tag{29}$$

follows, when $k\to 0$. Here, g_*, a_* are the values of the coupling constants g, a at the infrared-stable fixed point (26), respectively.

The energy pumping rate $\mathcal{E}$ is related to the correlation function of the random field f in the following fashion:

$$\mathcal{E}=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}\frac{D(k)}{k^2}. \tag{30}$$

For the correlation function (10) of the properly renormalized model it can readily be seen that for $\lambda>0$ $(\varepsilon<2)$, the energy pumping corresponding to the first term is concentrated at large wave numbers, as required for the assumed steady inverse energy cascade. The energy pumping due to the second term is always concentrated at large wave numbers. In contrast with the three-dimensional case, the inverse stirring length k_I serves as the upper cutoff for the inverse energy cascade. The choice of the lower cutoff m is different for the anticipated two scaling patterns.

When both the energy source and the enstrophy source are in the middle of the wave-number interval, as in most numerical simulations, dissipation scales are necessarily widely separated. The drag is responsible for the energy dissipation at low wave numbers and the microscale viscosity for the enstrophy dissipation at high wave numbers. On the other hand, when the sources are separated in the wave-number scale, as in atmospheric turbulence, dissipation due to both mechanisms is possible at wave numbers in between.

Therefore, when the energy source is at the upper end of the wave-number interval (k_I^e) and the enstrophy source at the lower end (k_I^b), I choose the borderline wave number $k_b=(g_*\xi_0^3/g_0\nu_0^3)^{1/(6-2\varepsilon)}$ as the lower cutoff m for the energy inertial range (and as the upper cutoff for the enstrophy inertial range). The self-similar enstrophy cascade is expected to take place for $k_I^b\ll k\ll k_b$ and the inverse energy cascade for $k_b\ll k\ll k_I^a$.

When the pumping of energy and enstrophy takes place in the middle of the wave-number interval (k_I) the natural choice for the upper cutoff of the enstrophy inertial range is the wave number of microscale dissipation $k_{\mathcal{B}}=\mathcal{B}^{1/6}\nu_0^{-1/2}$ and for the lower cutoff of the energy inertial range the wave number of drag dissipation $k_\xi=\mathcal{E}^{-1/2}\xi_0^{3/2}$.

The parameters g_0, a_0 of the model may be related to the energy pumping rate $\mathcal{E}$ in a manner similar to that used in the three-dimensional case [15]. For the correlation function (10) I obtain from Eq. (30) the relation

$$\mathcal{E}=\frac{a_0\nu_0^3}{16\pi}\Lambda^4+\frac{g_0\nu_0^3}{8\pi}\Lambda^{2(2-\varepsilon)}\frac{1}{2-\varepsilon}\left[\left(1+\frac{m^2}{\Lambda^2}\right)^{2-\varepsilon}-\frac{2-\varepsilon}{1-\varepsilon}\frac{m^2}{\Lambda^2}\left(1+\frac{m^2}{\Lambda^2}\right)^{1-\varepsilon}+\frac{1}{1-\varepsilon}\left(\frac{m^2}{\Lambda^2}\right)^{2-\varepsilon}\right], \quad (31)$$

where Λ is the upper cutoff parameter.

The coupling constant g_0 as a function of the energy pumping rate and cutoff parameters is substituted from the relation (31) in the asymptotic expression (29). The exponents in the power functions in Eq. (29) are exact in the ε expansion, but the ε expansion of the scaling function R is not simple. The one-loop calculation of the renormalization-group functions performed here allows one to find the scaling function at the leading order in the ε expansion. Therefore, only the leading order term of the coefficient of the power function $\Lambda^{2(2-\varepsilon)}$ in Eq. (31) may be consistently used when substituting g_0 from Eq. (31) in Eq. (29). At this accuracy the relation between $\mathcal{E}$ and g_0 is

$$\mathcal{E}=\frac{g_0\nu_0^3}{16\pi}\Lambda^{2(2-\varepsilon)}[1+O(\varepsilon)]+\frac{a_0\nu_0^3}{16\pi}\Lambda^4. \quad (32)$$

On the other hand, the connection between the stream function autocorrelation function and the energy spectrum $E(k)$, defined by the relation $\langle v^2(\mathbf{x})\rangle=2\int_0^\infty E(k)dk$, is

$$E(k)=\frac{k^3}{4\pi}W^{\mathrm{st}}_{\psi\psi R}(\mathbf{k}). \quad (33)$$

According to the asymptotic expression (29), this yields

$$E(k)=g_*^{1/3}g_0^{2/3}\nu_0^2\frac{k^{1-4\varepsilon/3}}{8\pi}R\left[1,g_*,a_*,\left(\frac{k_b}{k}\right)^{2-2\varepsilon/3},\frac{m}{k}\right]. \quad (34)$$

The right-hand side of Eq. (34) depends on several parameters, the values of which may be chosen to yield the expected universal scaling behavior in the inertial range. The spectrum (34) should be independent of the details of the energy pumping, i.e., independent of the upper cutoff Λ in the interval $m\ll k\ll\Lambda$. According to the relation (32), this goal is achieved by the choice $a_0=0$ and $\varepsilon=2$, as in the three-dimensional case. It should be borne in mind that the bare coupling constant a_0 is actually a book-keeping parameter reflecting the necessity of the introduction of the short-range term in the correlation function of the random force.

The spectrum (34) remains still dependent on the borderline wave number k_b and the small wave number cutoff parameter m. In the case of separated sources $m=k_b$, and it is sufficient to put $k_b=0$ to obtain completely scale-invariant energy spectrum. From the definition (28) for $\varepsilon=2$ it follows that

$$k_b=\left[\frac{\xi_0^3}{\nu_0^3}\frac{g_*}{g_0}\right]^{1/2}. \quad (35)$$

Therefore, to put $k_b=0$ is tantamount to putting $\xi_0=0$.

In the case of separated sinks the cutoff parameter $m=k_\xi$ and the relation of the two parameters k_ξ and k_b must be established. To this end, the expression (32) is substituted for $g_0\nu_0^3$ in Eq. (35). When the choice $a_0=0$ and the definition $k_\xi=\mathcal{E}^{-1/2}\xi_0^{3/2}$ are taken into account, it follows from Eqs. (32) and (35) that $k_b=(g_*/16\pi)^{1/2}k_\xi$. Thus, both parameters are proportional to $\xi_0^{3/2}$. Therefore, the completely scale-invariant energy spectrum may be obtained by putting $\xi_0=0$ also in the case of separated sinks.

Physically, the fact that $m=k_\xi\propto k_b$ leads to the somewhat unexpected conclusion that dissipation due to microscale viscosity is the dominant factor in dissipation in the (inverse) energy cascade; at the lower end of the energy inertial range dissipation due to microscale viscosity is of the same order of magnitude as dissipation due to drag. This effect is brought about by the renormalization, for in the unrenormalized vorticity equation both dissipative terms are of the same order of magnitude at wave numbers of the order

$$k_{0b}=\left(\frac{\xi_0}{\nu_0}\right)^{1/2}=\mathrm{Re}^{1/2}k_\xi\gg k_\xi, \quad (36)$$

since the Reynolds number of the energy inertial range $\mathrm{Re}=\mathcal{E}/\xi_0^2\nu_0$ is large in the asymptotic regime.

Thus, the choice $\xi_0=0$ renders the spectrum (34) completely scale invariant with the Kolmogorov exponents: $E\propto\mathcal{E}^{2/3}k^{-5/3}$. The coefficient may be calculated in the ε expansion. At the leading order the spectrum

$$E(k)=2^{4/3}3^{1/3}\mathcal{E}^{2/3}k^{-5/3} \quad (37)$$

follows, in which the only parameter is the energy pumping rate $\mathcal{E}$. The Kolmogorov constant C in the scaling law $E(k)=C\mathcal{E}^{2/3}k^{-5/3}$ is obtained in the leading order of the ε expansion from Eq. (37) as $C=2^{4/3}3^{1/3}=3.634$. This is less than the closure model prediction $C=6.69$ [10]. Most results of numerical simulations, which vary from $C=2.9$ [19] to $C\sim 14$ [20], also exceed the value obtained here. Experimental results [11] yield the range $3<C<7$.

The asymptotic behavior

$$E(k)=2(3\varepsilon)^{1/3}\mathcal{E}^{2/3}k^{1-4\varepsilon/3}k_I^{4(\varepsilon-2)/3},$$

resulting from Eq. (34) for $a_0=0$, $\xi_0=0$, is well justified in the ε expansion, in which it is safe to put the coefficient of friction $\xi_0=0$ and obtain a scaling regime independent of ξ_0 with pure power-law dependence on the wave number. The reason is that in the framework of perturbation theory, the scaling function R may be constructed in the form of an ε expansion at the fixed point of the RG:

$$R(1,g_*,a_*,z)=\sum_{n=0}^{\infty}\varepsilon^nR_n(z).$$

Inspection of the wave number and frequency integrals giving rise to this expansion reveals that the coefficients $R_n(z)$ are only weakly singular functions of the effective small wave-number cutoff $z\propto\xi_0$: they remain finite in the limit $z\to 0$, but contain singular terms of the type $z\ln z$.

For finite ε, however, it is not obvious that the self-similar behavior does not depend on ξ_0 and that the exponents of the powerlike asymptotics are those of Eq. (37), because for finite ε in the perturbation expansion there are always terms divergent in the limit $\xi_0 \to 0$.

A similar problem arises in the three-dimensional case. It has been thoroughly analyzed [15,21] with the aid of operator-product expansion. The operator-product expansion (fusion rule) of the product $\psi(t_1,\mathbf{x}_1)\psi(t_2,\mathbf{x}_2)$ of renormalized fields is an asymptotic expansion of the form

$$\psi(t_1,\mathbf{x}_1)\psi(t_2,\mathbf{x}_2)=\sum_n C_n(\tau,\mathbf{r})F_{nR}(t,\mathbf{x}) \tag{38}$$

valid in the correlation functions of the model. Here, $\tau=t_1-t_2\to 0$, $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2\to 0$, $t=(t_1+t_2)/2$, and $\mathbf{x}=(\mathbf{x}_1+\mathbf{x}_2)/2$. In Eq. (38) F_{nR} are scale-invariant linear combinations of renormalized composite operators, i.e., polynomial functions of the fields ψ and ψ' and their derivatives. The coefficient functions C_n are finite in the limit of $m\to 0$, therefore the behavior of the sum in the limit $m\to 0$ is determined by the scaling behavior of the composite operators.

Fusion rules such as Eq. (38) have been proved to hold for relativistic field theories (see, e.g., Ref. [18]), and are generally assumed to be true also in the case of first-order field theories. It has been shown [15,21] that asymptotic expressions obtained in the ε expansion for three-dimensional stochastic Navier-Stokes equation [1,2] are consistent for finite $0<\varepsilon<2$ in the limit $m\to 0$. This analysis involves calculation of anomalous dimensions of composite field operators with lowest canonical scaling dimensions with the subsequent resummation of those of them, which have negative total scaling dimensions [15].

This is a formidable task in three dimensions, and in two dimensions it becomes even worse, since the canonical dimensions of the stream function field ψ and the auxiliary field ψ' are equal, which leads to proliferation of relevant composite operators [16]. However, in the operator-product expansion of equal-time correlation functions of the stream function ψ there are no composite operators involving the auxiliary field ψ'. This can be seen by inspection of the graphs of the perturbation expansion, in which the equal-time correlation functions with the ψ' field always contain closed loops of the retarded $\psi'\psi$ propagator of the model and thus vanish. Since the analysis of Refs. [15,21] is independent of the space dimensionality, the results may be transferred to the two-dimensional case and thus the ε expansion of equal-time correlation functions is consistent in the limit $m\to 0$ for $0<\varepsilon<2$ also at two dimensions.

The enstrophy pumping rate $\mathcal{B}$ is related to the correlation function of the random field f in the following fashion:

$$\mathcal{B}=\frac{1}{2}\int\frac{d\mathbf{k}}{(2\pi)^2}D(k). \tag{39}$$

The enstrophy pumping is required to be concentrated at small wave numbers. For the IR-regularized correlation function (10) this corresponds to $\lambda<-1$ $(\varepsilon>3)$, and in the class of powerlike functions (when $m=0$) this corresponds to the limit $\lambda\to -1$ $(\varepsilon\to 3)$. The inverse stirring length (k_I or k_I^b) is the lower wave-number cutoff m for both separated sinks and separated sources, whereas the upper cutoff is the inverse dissipative length $k_{\mathcal{B}}=\mathcal{B}^{1/6}\nu_0^{-1/2}$.

Keeping again only the leading-order term of the ε expansion of the coefficient of the fractional power function, which emerges in the calculation of the integral (39) using the correlation function (10), I arrive at the following relation between the enstrophy pumping rate $\mathcal{B}$ and the coupling constant g_0:

$$\mathcal{B}=\frac{g_0\nu_0^3}{24\pi}k_{\mathcal{B}}^{2(3-\varepsilon)}[1+O(\varepsilon)]+\frac{a_0\nu_0^3}{24\pi}k_{\mathcal{B}}^6. \tag{40}$$

The relevant dimensional parameter in the enstrophy range is the enstrophy pumping rate $\mathcal{B}$, therefore dimensionless parameters in the asymptotic expression (34) should be chosen such that all other dimensional parameters are excluded from the energy spectrum.

From Eq. (40) it follows that the choice $\varepsilon=3$, $a_0=0$ renders the prefactor in the spectrum (34) independent of the microscale viscosity ν_0. It is a remarkable feature of the asymptotic spectrum (34) that in the limit $\varepsilon\to 3$ the power-function argument of the scaling function

$$\left(\frac{k_b}{k}\right)^{2-2\varepsilon/3}=\frac{g_*^{1/3}\xi_0}{g_0^{1/3}\nu_0}k^{-2+2\varepsilon/3}\to\frac{g_*^{1/3}\xi_0}{g_0^{1/3}\nu_0}$$

becomes independent of the wave number k.

Physically, this means that both microscale viscosity and drag contribute the same order of magnitude to the total dissipation in the enstrophy cascade, which again is quite unexpected from the point of view of the unrenormalized vorticity equation in the case of separated sinks. Moreover, in the case of separated sources the lower cutoff for the energy inertial range is $k_\xi=\mathcal{E}^{-1/2}\xi_0^{3/2}$ [the borderline wave number k_b (28) becomes meaningless for $\varepsilon=3$] and serves as a natural upper cutoff for the enstrophy inertial range, if a sink is assumed to exist. In this case, however, for the very existence of the enstrophy inertial range it would be necessary to require that $\xi_0>0$, which would rule out a spectrum with only one dimensional parameter $\mathcal{B}$. On the other hand, when both inertial ranges coexist, the natural borderline wave number is $k_0=(\mathcal{B}/\mathcal{E})^{1/2}$ and the enstrophy inertial range may well exist in the nondissipative limit.

In the limit $k\to 0$ I thus obtain the spectrum

$$E(k)=g_*^{1/3}g_0{}^{2/3}\nu_0^2\frac{k^{-3}}{8\pi}R\left[1,g_*,a_*,\frac{g_*{}^{1/3}\xi_0}{g_0^{1/3}\nu_0},\frac{m}{k}\right]. \tag{41}$$

Asymptotic behavior with only one parameter, the enstrophy pumping rate $\mathcal{B}$, may be obtained in the case of separated sinks by putting $\xi_0=m=0$ in this expression with the reservations made above for finite ε. In this case Eq. (40) leads to the following asymptotic expression for the energy spectrum in the enstrophy inertial range:

$$E(k)=2^{1/3}3^{4/3}\mathcal{B}^{2/3}k^{-3}. \tag{42}$$

This is the asymptotic form predicted from dimensional arguments [10]. The constant C' in the scaling law $E(k)=C'\mathcal{B}^{2/3}k^{-3}$ assumes the value $C'=2^{1/3}3^{4/3}=5.451$, which may be compared with the closure model prediction $C'=2.626$ [10].

It should be noted that the spectrum (41) becomes scale invariant already in the limit $m\to 0$ with finite coefficient of friction ξ_0. The corresponding leading-order expression is

$$E(k)=\frac{2^{1/3}3^{4/3}\mathcal{B}^{2/3}k^{-3}}{1+(4/3)^{2/3}\xi_0\mathcal{B}^{-1/3}} \tag{43}$$

with explicit dependence on the dimensionless combination $\xi_0\mathcal{B}^{-1/3}$.

However, for $\varepsilon\geqslant 2$ there are renormalized composite operators with negative scaling dimensions [15,21], e.g., the energy dissipation operator $\nabla^2\psi\nabla^2\psi$, whose overall contribution to the asymptotic behavior of the spectrum in the limit $m\to 0$ has not been determined. Therefore, in both cases (42) and (43) it remains an open question whether or not the scaling behavior of the spectrum is determined by the prefactor of the right-hand side of Eq. (41).

Since the coefficient of friction ξ_0 acts as a partial IR cutoff in the model, it seems plausible that the spectrum (41) with nonvanishing friction is less divergent in the limit $m\to 0$ than the frictionless spectrum. This would then imply that it is more plausible to arrive at friction-dependent scale-invariant spectrum for $\varepsilon=3$ [the leading order of which is given by Eq. (43)] than self-similar spectrum independent of the coefficient of friction.

V. CONCLUSION

In this work I have carried out the renormalization of the randomly forced vorticity equation with long-range correlated random force at two dimensions. It is shown that this equation, like the d-dimensional Navier-Stokes equation near two dimensions, cannot be consistently renormalized as such, but a local term has to be added to the correlation function of the random force to make the model multiplicatively renormalizable. Renormalization-group analysis of the asymptotic steady state of the modified model is carried out at the one-loop order for two different patterns of energy and enstrophy pumping.

When the energy pumping takes place at large wave numbers and enstrophy pumping at small wave numbers, both dissipative terms may be of the same order of magnitude at some intermediate wave-number scale, which serves as the lower cutoff in the energy inertial range and as the upper cutoff in the enstrophy range. The microscale viscosity is the dominant factor of dissipation in the inverse energy cascade. Thus, the asymptotic scaling behavior may be described by the stochastic vorticity equation without drag, which leads to the same Kolmogorov spectrum as in three dimensions (37) with the Kolmogorov constant $C=2^{4/3}3^{1/3}$.

The solution exhibits self-similar enstrophy inertial range either when the coefficient of friction is finite or when both inertial ranges coexist in the nondissipative limit. Therefore the energy spectrum, apart from the enstrophy pumping rate suggested by dimensional arguments, depends also on the coefficient of drag (but not viscosity), when the inertial intervals are separated by an energy and enstrophy sink. Moreover, in the renormalized model with finite dissipation the contribution to the enstrophy dissipation of both microscale viscosity and macroscale drag are of the same order of magnitude in the enstrophy cascade, contrary to the dominant role of the drag dissipation in the unrenormalized solution. At the leading order of the ε expansion for $\varepsilon=3$ the spectrum is a pure power function of the wave number (43), but the coefficient depends on both the enstrophy pumping rate $\mathcal{B}$ and coefficient of friction ξ_0.

When both energy and enstrophy pumping take place at the same wave-number scale, dissipation due to microscale viscosity dominates over dissipation due to friction in the inverse energy cascade also in this case. Therefore, quite unexpectedly, the asymptotic self-similar behavior in the energy inertial range is described by the stochastic vorticity equation without friction for both patterns of pumping. The enstrophy inertial range may well exist also for vanishing friction in this case. In the nondissipative limit the renormalization-group equations then seem to predict self-similar behavior with the energy spectrum (42), which is in accord with the prediction [10] based on dimensional arguments.

APPENDIX: CALCULATION OF THE RENORMALIZATION CONSTANTS

The bare propagators corresponding to the action (11) with the Pauli-Villars regularization in the time-wave-vector representation are

$$\Delta_{\psi\psi'}(t,\mathbf{k})=\theta(t)\,\frac{e^{-[\nu k^2+\xi_0]t}}{k^2},$$

$$\Delta_{\psi\psi}(t,\mathbf{k})=\frac{g\nu^3M^{2\varepsilon}(k^2+m^2)^{-\varepsilon}+a\nu^3\Lambda^2(\Lambda^2+k^2)^{-1}}{2(\nu k^2+\xi_0)}\times e^{-[\nu k^2+\xi_0]|t|},$$

where $\theta(t)$ is the step function. The renormalization constant Z_1 is found from the requirement that the renormalized 1PI Green function

$$\Gamma_{\psi'\psi'R}(\omega,\mathbf{k})=g\nu^3M^{2\varepsilon}\frac{k^4}{(k^2+m^2)^{\varepsilon}}+a\nu^3Z_1\frac{k^4\Lambda^2}{k^2+\Lambda^2}+\Sigma_{\psi'\psi'}(\omega,\mathbf{k}) \tag{A1}$$

is finite, when $\varepsilon\to 0$ and $\Lambda\to\infty$. In order to distinguish between the terms with short-range and long-range correlations, the inverse correlation length m must be set equal to zero before the calculation of the renormalization constants. This does not cause problems in the ε expansion, but requires separate analysis for finite ε. The self-energy term $\Sigma_{\psi'\psi'}$ at the one-loop order is given by the integral

$$\Sigma^{(1)}_{\psi'\psi'}(t,\mathbf{k})=\tfrac{1}{2}(\epsilon_{in}\epsilon_{i'n'}k_mk_ik_{m'}k_{i'}+\epsilon_{in}\epsilon_{i'm'}k_mk_ik_{n'}k_{i'}+\epsilon_{im}\epsilon_{i'n'}k_nk_ik_{m'}k_{i'}+\epsilon_{im}\epsilon_{i'm'}k_nk_ik_{n'}k_{i'})\times\int\frac{d\mathbf{q}}{(2\pi)^2}(k-q)_m(k-q)_{m'}q_nq_{n'}\times\Delta_{\psi\psi}(t,\mathbf{k}-\mathbf{q})\Delta_{\psi\psi}(t,\mathbf{q}). \tag{A2}$$

According to Eq. (A1), the renormalization constant Z_1 is found from the representation

$$\int dt\Sigma^{(1)}_{\psi'\psi'}(t,\mathbf{k})=-a\nu^3(Z_1-1)k^4+o(k^4)$$

in the limit $\varepsilon\to 0$ and $\Lambda\to\infty$. Differentiation with respect to m in Eq. (A2) leads to UV convergent integral, which means that Z_1 is independent of m at one-loop level. This is actually the case in the MS scheme to all orders [18]. Thus, the calculation of the renormalization constants may be simplified by choosing $m=0$.

Similarly, differentiation of the integral (A2) yields UV convergent integral, therefore I may put $\mathbf{k}=0$ in it for the calculation of the renormalization constants. Note that the power k^4 is already factorized in Eq. (A2). The integral with vanishing external wave vector may be expressed as a sum of products of Kronecker symbols with scalar coefficients. When the contractions of indices are carried out, these coefficients are expressed in terms of scalar integrals. As a result, the expression

$$\Sigma^{(1)}_{\psi'\psi'}(t,\mathbf{k})=\frac{k^4}{4}\int\frac{d\mathbf{q}}{(2\pi)^2}\frac{q^4e^{-2[\nu q^2+\xi_0]|t|}}{4(\nu q^2+\xi_0)^2}\left[(g\nu^3M^{2\varepsilon})^2q^{-4\varepsilon}+2g\nu^3M^{2\varepsilon}a\nu^3q^{-2\varepsilon}\frac{\Lambda^2}{q^2+\Lambda^2}+(a\nu^3)^2\frac{\Lambda^4}{(q^2+\Lambda^2)^2}\right] \tag{A3}$$

follows. The "mass" parameter m has been set equal to zero here to simplify the calculation. To extract the divergent part, it is sufficient to calculate the self-energy at zero frequency, which amounts to integration of Eq. (A3) over the time variable. Calculation of the resulting integrals over the wave vector $\mathbf{q}$ is straightforward and leads to the result

$$\begin{aligned}
&\Sigma^{(1)}_{\psi'\psi'}(\omega=0,\mathbf{k}\to 0)\\
&=\frac{k^4}{16(2\pi)^2\nu^3}\Bigg\{(g\nu^3M^{2\varepsilon})^2\frac{\pi\Gamma(2\varepsilon)\Gamma(3-2\varepsilon)}{2(\xi_0/\nu)^{2\varepsilon}}+2g\nu^3M^{2\varepsilon}a\nu^3\Lambda^2\frac{\pi\Gamma(-1+\varepsilon)\Gamma(3-\varepsilon)}{2(2-\varepsilon)}\\
&\times\left[2\frac{(\xi_0/\nu)^{2-\varepsilon}-\Lambda^{2(2-\varepsilon)}}{(\xi_0/\nu-\Lambda^2)^3}-2\frac{(2-\varepsilon)(\xi_0/\nu)^{1-\varepsilon}}{(\xi_0/\nu-\Lambda^2)^2}+(2-\varepsilon)(1-\varepsilon)\frac{(\xi_0/\nu)^{-\varepsilon}}{\xi_0/\nu-\Lambda^2}\right]-(a\nu^3)^2\Lambda^4\pi\\
&\times\left[3\frac{(\xi_0/\nu)^2\ln(\xi_0/\nu)-\Lambda^4\ln\Lambda^2}{(\xi_0/\nu-\Lambda^2)^4}-\frac{4(\xi_0/\nu)\ln(\xi_0/\nu)+2\Lambda^2\ln\Lambda^2+2(\xi_0/\nu)^2+\Lambda^2}{(\xi_0/\nu-\Lambda^2)^3}+\frac{\ln(\xi_0/\nu)+3/2}{(\xi_0/\nu-\Lambda^2)^2}\right]\Bigg\}.
\end{aligned}$$

The renormalization constant Z_1 is obtained from the leading singular terms of this expression in the limit $\varepsilon\to 0$ and $\Lambda\to\infty$. The result depends on the order of passing to the limit and yields the first expression in Eq. (14). The calculation of the renormalization constant from the self-energy term $\Sigma_{\psi\psi'}$ is similar, but no ambiguity in the limit $\varepsilon\to 0$ and $\Lambda\to\infty$ arises there.

[1] C. De Dominicis and P.C. Martin, Phys. Rev. A **19**, 419 (1979).

[2] L.Ts. Adzhemyan, Yu.M. Pis'mak, and A.N. Vasil'ev, Teor. Mat. Fiz. **57**, 268 (1983) [Theor. Math. Phys. **57**, 1131 (1983)].

[3] V. Yakhot and S.A. Orszag, Phys. Rev. Lett. **57**, 1722 (1986); J. Sci. Comput. **1**, 3 (1986).

[4] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasil'ev, Usp. Fiz. Nauk **166**, 1257 (1996) [Phys. Usp. **39**, 1193 (1996)].

[5] A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics: Mechanics of Turbulence, Vol. 2* (MIT Press, Cambridge, MA, 1975).

[6] P. Olla, Phys. Rev. Lett. **67**, 2465 (1991); Int. J. Mod. Phys. B **8**, 581 (1994).

[7] N.V. Antonov, Teor. Mat. Fiz. **112**, 417 (1997).

[8] M.F. Larsen, M.C. Kelley, and K.S. Gage, J. Atmos. Sci. **39**, 1035 (1982).

[9] M.E. Maltrud and G.K. Vallis, J. Fluid Mech. **228**, 321 (1991); A. Babiano, B. Dubrulle, and P. Frick, Phys. Rev. E **52**, 3719 (1995).

[10] R.H. Kraichnan, Phys. Fluids **10**, 1417 (1967); J. Fluid Mech. **47**, 525 (1971).

[11] J. Sommeria, J. Fluid Mech. **170**, 139 (1986).

[12] V. Borue, Phys. Rev. Lett. **72**, 1475 (1994).

[13] D. Ronis, Phys. Rev. A **36**, 3322 (1987).

[14] J.-D. Fournier, P.-L. Sulem, and A. Pouquet, J. Phys. A **15**, 1393 (1982); L.Ts. Adzhemyan, A.N. Vasil'ev, and M. Hnatic, Teor. Mat. Fiz. **64**, 196 (1985) [Theor. Math. Phys. **64**, 777 (1985)].

[15] L.Ts. Adzhemyan, N.V. Antonov, and A.N. Vasil'ev, Zh. Eksp. Teor. Fiz. **95**, 1272 (1989) [Sov. Phys. JETP **68**, 733 (1989)].

[16] J. Honkonen and M.Yu. Nalimov, Z. Phys. B **99**, 297 (1996).

[17] D. Forster, D.R. Nelson, and M.J. Stephen, Phys. Rev. A **16**, 732 (1977).

[18] J. Collins, *Renormalization* (Cambridge University Press, Cambridge, 1985).

[19] J.R. Herring and J.C. McWilliams, J. Fluid Mech. **153**, 229 (1985).

[20] E.D. Siggia and H. Aref, Phys. Fluids **24**, 171 (1981).

[21] L.Ts. Adzhemyan, N.V. Antonov, and T.L. Kim, Teor. Mat. Fiz. **100**, 382 (1994) [Theor. Math. Phys. **100**, 1086 (1994)].